Thus, the simultaneously limiting states are attained in all cross sections of the beam only under the action of a concentrated force applied at the free end of the beam. Therefore, for the problem under consideration (in contrast to the problem discussed in Sect. 4), there exists the worst force in the class F, for which the optimal solution obtained with regard solely to that force, is also optimal for the entire class as a whole.

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AXISYMMETRIC CONTACT PROBLEM FOR AN ELASTIC INHOMOGENEOUS HALF-SPACE IN THE PRESENCE OF COHESION

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There is obtained the exact solution of the axisymmetric contact problem on the indentation of a circular punch into an elastic half-space having a variable modulus of elasticity $E = E_\nu z^\nu$ $(0 \leqslant \nu < 1)$ in the case of the presence of complete cohesion.

1. For the formulation of the axisymmetric contact problem on the indentation of a circular punch into any linearly-deformable foundation, obviously, it is sufficient to know the vertical and radial displacements of the surface points of the foundation due to the action of vertical and radial loads of the form

$$p_0(r) = \delta(r-\rho), \qquad q_0(r) = \delta(r-\rho) \qquad (r,\ \rho \geqslant 0) \tag{1.1}$$

where $\delta(x)$ is Dirac's impulse function, describing in this case a concentrated load along a circumference (of radius ρ).

We adopt the following rule for the signs of the loads and displacements. The vertical loads and the corresponding displacements are considered to be positive if they are oriented downwards while the radial load and displacement is positive if they are oriented in the direction in which the value of r increases.

Assume that the vertical displacements $\theta_0{}^*\rho K_{00}(r, \rho)$ and the radial displacements $-\theta_1{}^*\rho K_{10}(r, \rho)$ are known in terms of the load $p_0(r)$ and that the vertical displacements $-\theta_1{}^*\rho K_{01}(r, \rho)$ and the radial displacements $\theta_2{}^*\rho K_{11}(r, \rho)$ are known in terms of the load $q_0(r)$. (Here the signs for displacements are taken as in the case of the classical foundation of a homogeneous isotropic half-space).

Then, denoting the unknown normal and tangential contact stresses under the circular punch (of radius a) by $p(r)$ and $q(r)$, respectively, we arrive at the following system of integral equations:

$$\theta_0{}^* \int_0^a K_{00}(r, \rho)\rho p(\rho)\,d\rho - \theta_1{}^* \int_0^a K_{01}(r, \rho)\rho q(\rho)\,d\rho = f(r) \qquad (1.2)$$

$$\theta_2{}^* \int_0^a K_{11}(r, \rho)\rho q(\rho)\,d\rho - \theta_1{}^* \int_0^a K_{10}(r, \rho)\rho p(\rho)\,d\rho = g(r)$$

Here $f(r)$ and $g(r)$ are given functions (defining the displacements in the contact zone), where the first is given, up to an additive constant (penetration of the punch), while the second vanishes at zero.

We note that in the case of the classical foundation the influence function K_{mn} is determined by the formulas ($J_n(z)$ is Bessel's function)

$$K_{mn}(r, \rho) = W_{mn}^0(r, \rho), \qquad \theta_0{}^* = \theta_2{}^* = 2E^{-1}(1-\mu^2) \qquad (1.3)$$

$$\theta_1{}^* = (1-2\mu)(1+\mu)E^{-1}, \qquad W_{mn}^\nu(r, \rho) = \int_0^\infty t^\nu J_m(tr) J_n(t\rho)\,dt$$

Let us consider next the construction of the influence function for the half-space with a modulus of elasticity of the form $E = E_\nu z^\nu$. We make use of the fact that in the plane problem the analogous function can be easily constructed (see results of [1, 2]). As a result we find that under the action of a vertical (downward) load $p(x)$ on the boundary of a half-plane, with the indicated modulus of elasticity, the verical displacements $w_p(x)$ and the horizontal displacements $u_p(x)$ (the x-axis is oriented to the right) of the boundary points of the half-plane can be represented in the form

$$w_p(x) = \theta_0 \int_{-\infty}^{\infty} \frac{p(y)\,dy}{\nu\,|x-y|^\nu}, \qquad u_p(x) = -\theta_1 \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)\,p(y)\,dy}{|x-y|^\nu} \qquad (1.4)$$

$$\theta_0 = \frac{(1-\mu^2)\gamma C_\nu \sin {}^1/_2\gamma\pi}{(1+\nu)E_\nu}, \qquad \theta_1 = -\frac{(1-\mu^2)C_\nu \cos {}^1/_2\gamma\pi}{\nu E_\nu}$$

$$C_\nu = \frac{2^{\nu+1}\Gamma[{}^1/_2(\nu+\gamma+3)]\,\Gamma[{}^1/_2(\nu-\gamma+3)]}{\pi\Gamma(\nu+2)}, \quad \gamma^2 = (1+\nu)\left(1-\frac{\nu\mu}{1-\mu}\right)$$

Under the action of a horizontal (from left to right) load $q(x)$ on the boundary of the half-plane, the corresponding displacements have the form

$$w_q(x) = \theta_1 \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)\,q(y)\,dy}{|x-y|^\nu}, \qquad u_q(x) = \theta_2 \int_{-\infty}^{\infty} \frac{q(y)\,dy}{\nu\,|x-y|^\nu} \qquad (1.5)$$

$$\theta_2\gamma E_\nu = (1-\mu^2)(1+\nu)C_\nu \sin {}^1/_2\gamma\pi$$

The given formulas (1.4), (1.5) allow us to construct the required influence functions. To this end, we use the transition formulas from plane to axisymmetric states given in [3]. In conformity with the displacements of the surface points they have the form

$$w^\circ(r) = -\frac{1}{\pi}\int_r^\infty \frac{dw^*(x)}{\sqrt{x^2-r^2}}, \qquad u^\circ(r) = -\frac{1}{\pi}\int_r^\infty \frac{x\,du^*(x)}{r\sqrt{x^2-r^2}} \tag{1.6}$$

Here $w^\circ(r)$ and $u^\circ(r)$ are the vertical and the radial displacements of the foundation surface points in the axisymmetric state, while $w^*(x)$ and $u^*(x)$ are the vertical and the horizontal displacements of the corresponding plane state.

We denote, for example, by $u_{10}^\circ(r)$ the radial displacements of the foundation surface points due to the load $p_0(r)$. According to [3], the corresponding horizontal displacements $u_{10}^*(x)$ of the plane state, taking into account (1.4), are determined by the formula

$$u_{10}^*(x) = -2\rho\theta_1\int_{-\rho}^{\rho} \frac{\operatorname{sgn}(x-y)\,dy}{|x-y|^\nu\sqrt{\rho^2-y^2}} \tag{1.7}$$

Reducing here the integration to the interval $(0, \rho)$ and using the formula 3.762(2) of [4], after interchanging the order of integration and differentiation we obtain

$$\frac{du_{10}^*(x)}{dx} = -\frac{2\rho\pi\theta_1}{\Gamma(\nu)\sin{}^1/_2\nu\pi}\int_0^\infty t^\nu\cos xtJ_0(\rho t)\,dt = \tag{1.8}$$

$$\frac{2\rho\pi\theta_1}{x\Gamma(\nu)\sin{}^1/_2\nu\pi}\int_0^\infty \sin xt\,d\,[t^\nu J_0(\rho t)]$$

The second equality is obtained by integrating by parts. Substituting (1.8) into the second formula of (1.6), after interchanging the order of integration, using the formula 3.753(3) of [4] and integrating by parts, we find

$$u_{10}^\circ(r) = -\theta_1\pi\rho\,[\sin{}^1/_2\nu\pi\Gamma(\nu)]^{-1}W_{10}^\nu(r,\rho) \tag{1.9}$$

Proceeding in the same way for the determination of the other displacements, we find that regarding the half-space with $E = E_\nu z^\nu$, the influence functions and the parameters contained in (1.2) have the form

$$K_{mn}(r,\rho) = W_{mn}^\nu(r,\rho), \qquad \theta_1^* = \pi\theta_1\,[\sin{}^1/_2\nu\pi\Gamma(\nu)]^{-1} \tag{1.10}$$
$$\theta_{0,2}^* = \pi\theta_{0,2}\,[\cos{}^1/_2\nu\pi\Gamma(\nu+1)]^{-1}$$

i.e. as in the case of the classical foundation, the influence functions are expressed in terms of the Weber-Sonin integrals (1.3).

2. We reduce the system of integral equations (1.2) corresponding to the case (1.10) to a unique integral equation admitting an exact solution. We consider the integrals

$$P(t) = \int_0^a J_0(t\rho)\rho p(\rho)\,d\rho, \qquad Q(t) = \int_0^a J_1(t\rho)\rho q(\rho)\,d\rho \tag{2.1}$$

If we make use of the representations [4]

$$J_0(t\rho) = \frac{2}{\pi}\int_0^\rho \frac{\cos ts\,ds}{\sqrt{\rho^2-s^2}}, \qquad \rho J_1(t\rho) = \frac{2}{\pi}\int_0^\rho \frac{s\sin ts\,ds}{\sqrt{\rho^2-s^2}} \tag{2.2}$$

then, after interchanging the order of integration, instead of (2.2) we have

$$P(t)=\int_0^a \sigma(s)\cos ts\,ds, \qquad Q(t)=\int_0^a \tau(s)\sin ts\,ds \tag{2.3}$$

$$\sigma(s)=\frac{2}{\pi}\int_s^a \frac{\rho p(\rho)\,d\rho}{\sqrt{\rho^2-s^2}}, \qquad \tau(s)=\frac{2s}{\pi}\int_s^a \frac{q(\rho)\,d\rho}{\sqrt{\rho^2-s^2}} \tag{2.4}$$

The functions $\sigma(s)$ and $\tau(s)$ will be continued, by necessity, for negative values of the argument as even and odd functions, respectively, i.e.

$$\sigma(-s)=\sigma(s), \qquad \tau(-s)=-\tau(s) \tag{2.5}$$

Taking into account (1.10) and (1.3), we interchange in (1.2) the order of integration and we make use of (2.1), (2.3), (2.4). Then, to the first and the second of the equations (1.2) we apply, respectively, the operators

$$I^{\cdot}[\varphi(x)]=\int_0^x \frac{\varphi(r)\,r\,dr}{\sqrt{x^2-r^2}}, \qquad J^{\cdot}[\varphi(x)]=\frac{d}{dx}\int_0^x \frac{y\,dy}{\sqrt{x^2-y^2}}\int_0^y \varphi(r)\,dr \tag{2.6}$$

The subsequent use of the formulas of Sonin [4]

$$\int_0^x \frac{J_0(tr)\,r\,dr}{\sqrt{x^2-r^2}}=\frac{\sin xt}{t}, \quad \frac{d}{dx}\int_0^x \frac{J_0(tr)-1}{\sqrt{x^2-r^2}}\,r\,dr=\cos xt-1 \tag{2.7}$$

reduces Eqs. (1.2) to the form

$$\theta_2^*\int_0^a \sigma(s)\,ds\int_0^\infty \frac{\sin tx\cos ts}{t^{1-\nu}}\,dt-\theta_1^*\int_0^a \tau(s)\,ds\int_0^\infty \frac{\sin tx\sin ts}{t^{1-\nu}}\,dt=I^{\cdot}[f(x)] \tag{2.8}$$

$$-\theta_2^*\int_0^a \tau(s)\,ds\int_0^\infty \frac{(\cos tx-1)\sin ts}{t^{1-\nu}}\,dt+$$
$$\theta_1^*\int_0^a \sigma(s)\,ds\int_0^\infty \frac{(\cos tx-1)\cos ts}{t^{1-\nu}}\,dt=J^{\cdot}[g(x)]$$

Using formulas 3.762, 3.761(4), 3.761(9) of [4], of the assumption (2.5) as well as of the formulas (1.10), (1.4) and (1.5), instead of (2.8) we have

$$\int_{-a}^a \frac{\operatorname{sgn}(x-s)\,\sigma(s)\,ds}{|x-s|^\nu}-\frac{\operatorname{tg}{}^1/_2\lambda\pi}{\varkappa(\operatorname{tg}{}^1/_2\nu\pi)^2}\int_{-a}^a \frac{\tau(s)\,ds}{|x-s|^\nu}= \tag{2.9}$$
$$=\frac{2\varkappa^{-1}\operatorname{tg}{}^1/_2\lambda\pi}{\pi\theta_1\operatorname{tg}{}^1/_2\nu\pi}\int_0^x \frac{f(r)\,r\,dr}{\sqrt{x^2-r^2}}$$

$$\int_{-a}^a \frac{\operatorname{sgn}(x-s)\,\tau(s)\,ds}{|x-s|^\nu}+\frac{\varkappa\operatorname{tg}{}^1/_2\lambda\pi}{(\operatorname{tg}{}^1/_2\nu\pi)^2}\int_{-a}^a \frac{\sigma(s)\,ds}{|x-s|^\nu}=\int_{-a}^a \frac{\operatorname{sgn}(-s)\,\tau(s)\,ds}{|s|^\nu}+$$
$$\frac{\varkappa\operatorname{tg}{}^1/_2\lambda\pi}{(\operatorname{tg}{}^1/_2\nu\pi)^2}\int_{-a}^a \frac{\sigma(s)\,ds}{|s|^\nu}+\frac{2\varkappa\operatorname{tg}{}^1/_2\lambda\pi}{\pi\theta_1\operatorname{tg}{}^1/_2\nu\pi}\int_0^x \frac{xg(r)\,dr}{\sqrt{x^2-r^2}}$$
$$(\lambda=\gamma-1,\ \varkappa=(1+\lambda)(1+\nu)^{-1})$$

Multiplying the first equation of (2.9) by $\sqrt{\varkappa}$, the second one by $i\varkappa^{-1/2}$ and adding the results, we obtain the unique integral equation

$$\int_{-a}^{a}[k_\nu(x-s)-k_\nu(-s)]\,\chi(s)\,ds=F(x) \tag{2.10}$$

relative to the complex function

$$\chi(s)=\sqrt{\varkappa}\sigma(s)+\varkappa^{-1/2}i\tau(s) \tag{2.11}$$

The kernel and the right-hand side of the equation (2.10) are defined by the formulas

$$k_\nu(y)=|y|^{-\nu}[\operatorname{sgn}y+i\operatorname{tg}{}^1/_2\lambda\pi\,(\operatorname{ctg}{}^1/_2\nu\pi)^2] \tag{2.12}$$

$$F(x)=\frac{2\operatorname{tg}{}^1/_2\lambda\pi x}{\pi\theta_1\operatorname{tg}{}^1/_2\nu\pi\sqrt{\varkappa}}\int_0^1\frac{[f(x\rho)\rho+i\varkappa g(x\rho)]\,d\rho}{\sqrt{1-\rho^2}}$$

3. The solution of the obtained integral equation can be constructed in an explicit form by the method of reduction to the Riemann boundary value problem ([5], p. 603 of Russian edition). However it is more convenient to make use of the following spectral relation for the Jacobi polynomials $P_m^{\alpha,\beta}(x)$:

$$\frac{d^{k+1}}{dx^{k+1}}\int_{-a}^{a}\left[\operatorname{sgn}(x-t)+\frac{\operatorname{tg}\pi(\alpha+{}^1/_2\nu)}{\operatorname{tg}{}^1/_2\nu\pi}\right]\frac{(a+t)^{k+\nu+\alpha}P_m^{-\alpha,\,k+\nu+\alpha}(t/a)}{(a-t)^{\alpha}|x-t|^{\nu}}\,dt= \tag{3.1}$$

$$\frac{\pi\Gamma(m+\nu+k+1)\,P_m^{k+\nu+\alpha,-\alpha}(x/a)}{\Gamma(\nu)\sin{}^1/_2\nu\pi\cos\pi(\alpha+{}^1/_2\nu)\,m!}$$

$$(0\leqslant\operatorname{Re}\nu<1,\ \operatorname{Re}\alpha>-1,\ k=0,1,2,\ldots)$$

This relation is the generalization of the spectral relation given in [6] to the case $\nu\neq 0$ and is proved by the same method.

In order to apply the spectral relation (3.1), we differentiate both sides of Eq.(2.10)

$$\frac{d}{dx}\int_{-a}^{a}\left[\operatorname{sgn}(x-s)+\frac{i\operatorname{tg}{}^1/_2\lambda\pi}{(\operatorname{tg}{}^1/_2\nu\pi)^2}\right]\frac{\chi(s)}{|x-s|^{\nu}}\,ds=F'(x) \tag{3.2}$$

Then, from the equation $\operatorname{tg}\pi(\alpha+{}^1/_2\nu)=i\operatorname{tg}{}^1/_2\lambda\pi\operatorname{ctg}{}^1/_2\nu\pi$ we find that

$$\alpha=i\beta-\frac{\nu}{2},\qquad \beta=\frac{1}{2\pi}\ln\frac{\sin{}^1/_2(\nu+\lambda)\pi}{\sin{}^1/_2(\nu-\lambda)\pi} \tag{3.3}$$

We can show that $\nu\geqslant\lambda,\ \nu+\lambda<2$.

The existence of the spectral relation (3.1) allows us to apply the method of orthogonal polynomials [7] in order to obtain the exact solution of the equation (3.2). As a result we obtain

$$\chi(x)=\sum_{m=0}^{\infty}\frac{\chi_m P_m^{\beta}(x)}{\varphi_\beta(x)},\qquad \varphi_\beta(x)=\frac{(a-x)^{i\beta}(a+x)^{-i\beta}}{(a^2-x^2)^{\nu/2}} \tag{3.4}$$

$$(P_m^{\beta}(x)=P_m^{{}^1/_2\nu-i\beta,\,{}^1/_2\nu+i\beta}(x/a))$$

$$\chi_n=\frac{n!^2(\nu+1+2n)\,\Gamma(\nu)\sin{}^1/_2\nu\pi\operatorname{ch}\pi\beta F_r}{\pi(2a)^{1+\nu}|\Gamma(1+n+{}^1/_2\nu+i\beta)|^{2}} \tag{3.5}$$

$$F_n = \int_{-a}^{a} \frac{F'(x) P_n^{-\beta}(x)\, dx}{\varphi_{-\beta}(x)} \tag{3.6}$$

The desired contact stresses under the punch, according to (2.4) and (2.11), are determined by the formulas

$$rp(r) = -\frac{1}{\sqrt{\varkappa}} \frac{d}{dr} \int_{r}^{a} \frac{s \operatorname{Re}[\chi(s)]\, ds}{\sqrt{s^2 - r^2}} \tag{3.7}$$

$$q(r) = -\sqrt{\varkappa} \frac{d}{dr} \int_{r}^{a} \frac{\operatorname{Im}[\chi(s)]\, ds}{\sqrt{s^2 - r^2}}$$

where, for the separation of the imaginary and real parts in (3.4), we can use the formula

$$\frac{P_m^{\beta}(x)}{\varphi_\beta(x)} = \frac{(-1)^m}{(2a)^m m!} \left\{ \frac{d^m}{dx^m} \left[(a^2 - x^2)^{1/2\nu + m} \cos\left(\beta \ln \frac{a+x}{a-x}\right)\right] + i \frac{d^m}{dx^m} \left[(a^2 - x^2)^{1/2\nu + m} \sin\left(\beta \ln \frac{a+x}{a-x}\right)\right]\right\}$$

Taking into account the formulas (2.1), (2.3), (2.5) and (2.11), we find that the force which compresses the punch is

$$P = 2\pi \int_0^a rp(r)\, dr = \pi \int_{-a}^{a} \sigma(s)\, ds = \frac{\pi}{\sqrt{\varkappa}} \operatorname{Re}\left[\int_{-a}^{a} \chi(s)\, ds\right]$$

Substituting the series (3.4) into the last formula and using the orthogonality of Jacobi polynomials, we obtain

$$P = \varkappa^{-1/2} \nu^{-1} \sin {}^1/_2 \nu\pi \operatorname{ch} \pi\beta \operatorname{Re}[F_0] \tag{3.8}$$

4. In conclusion we consider an important particular case of the problem under consideration, when the base of the punch is plane and the punch is loaded by a central force P. We denote the desired magnitude of the penetration of the punch by δ. In this case the right-hand sides of the system (1.2) become essentially simpler

$$f(r) = \delta, \qquad g(r) \equiv 0 \tag{4.1}$$

which, in turn, leads to the simplification of the formulas (2.12) and (3.6)

$$F'(x) = \frac{2 \operatorname{tg} {}^1/_2 \lambda\pi\delta}{\pi\theta_1 \operatorname{tg} {}^1/_2 \nu\pi \sqrt{\varkappa}}, \qquad F_0 = \frac{2 |\Gamma(1 + {}^1/_2\nu + i\beta)|^2 \operatorname{tg} {}^1/_2 \lambda\pi\delta}{\pi (2a)^{-1-\nu} \Gamma(\nu + 2) \theta_1 \sqrt{\varkappa} \operatorname{tg} {}^1/_2 \nu\pi} \tag{4.2}$$

Substituting the last expression into (3.8), we find the penetration of the punch

$$\delta = \frac{P\varkappa\pi (1 - \mu^2) \Gamma(\nu + 2) C_\nu \cos {}^1/_2 \lambda\pi}{2E_\nu (2a)^{1+\nu} |\Gamma(1 + {}^1/_2\nu + i\beta)|^2 \operatorname{ch} \pi\beta \cos {}^1/_2 \nu\pi} \tag{4.3}$$

The formulas for the contact stresses (3.7) also become essentially simpler since by virtue of (4.2) and the orthogonality of Jacobi polynomials, we have $F_n = 0$, $n = 1, 2, \ldots$ and in the series (3.4) only the first term remains, i.e.

$$\chi(s) = \chi_0 (a^2 - s^2)^{\nu/2} (a - s)^{-i\beta} (a + s)^{i\beta} \tag{4.4}$$

Substituting this expression into (3.7) and using (3.5), (4.2) and (4.3), we obtain for the case under consideration the following formulas for the contact stresses:

$$rp(r) = -\frac{P\Gamma(\nu+2)}{(2a)^{1+\nu}\,|\,\Gamma(1+{}^1/_2\nu+i\beta)\,|^2}\,\frac{d}{dr}\int\limits_r^a \frac{s\,(a^2-s^2)^{{}^1/_2\nu}}{\sqrt{s^2-r^2}}\times \tag{4.5}$$
$$\cos\left(\beta\ln\frac{a+s}{a-s}\right)ds$$
$$q(r) = -\frac{\varkappa P\Gamma(\nu+2)}{(2a)^{1+\nu}\,|\,\Gamma(1+{}^1/_2\nu+i\beta)\,|^2}\,\frac{d}{dr}\int\limits_r^a \frac{(a^2-s^2)^{{}^1/_2\nu}}{\sqrt{s^2-r^2}}\times$$
$$\sin\left(\beta\ln\frac{a+s}{a-s}\right)ds$$

Setting $\nu = 0$ in the above formulas, we obtain the corresponding results for the classical foundation. The formula (4.3) takes the form obtained in [8, 9]. As far as the formulas (4.5) for the contact stresses are concerned, they (for $\nu = 0$) take the form different in structure from those derived in [8, 9]. In order to arrive at the formulas given in [8], it is necessary to perform a series of preliminary nontrivial operations (these operations conceptually are similar to those carried out in paper [10] in solving the analogous problem for a classical foundation).

These operations consist in the following. We introduce the functions

$$p^*(x) = \int\limits_x^a rp(r)\,dr, \qquad q^*(x) = \int\limits_x^a q(r)\,dr \tag{4.6}$$

and we integrate both formulas (3.7) along the interval (x, a), after which we apply the operator I defined by (2.6). As a result, instead of (3.7) (taking into account (2.11)) we have

$$I[p^*(x)] = \int\limits_0^a l(x,s)\,s\sigma(s)\,ds, \quad I[q^*(x)] = \int\limits_0^a l(x,s)\,\tau(s)\,ds \tag{4.7}$$
$$l(x,s) = \int\limits_0^{\min(x,s)} \frac{u\,du}{[(x^2-u^2)(s^2-u^2)]^{{}^1/_2}} = \frac{1}{2}\ln\left|\frac{x+s}{x-s}\right|$$

Taking into account (2.5), we replace the integration interval in the relations (4.7) by the interval $(-a, a)$, after which we differentiate them. As a result we obtain

$$\frac{d}{dx}I[p^*(x)] = \frac{1}{2}\Gamma[x\sigma(x)], \quad \frac{d}{dx}I[q^*(x)] = \frac{1}{2}\Gamma[\tau(x)] \tag{4.8}$$
$$\Gamma(\varphi) = \int\limits_{-a}^a \frac{\varphi(s)\,ds}{s-x}$$

where Γ is the finite Hilbert transform.

Then, taking into account the identity

$$s(s-x)^{-1} = 1 + x(s-x)^{-1}$$

we write the first relation of (4.8) in the form

$$\frac{d}{dx}I[p^*(x)] = \frac{x}{2}\Gamma[\sigma(x)] + \frac{1}{2}\int\limits_{-a}^a \sigma(s)\,ds$$

Applying to this and to the second relation of (4.8) Abel's transformation formulas and taking into account (4.6), we obtain

$$\sqrt{\varkappa}\,p(r) + i\varkappa^{-{}^1/_2}q(r) = -\frac{1}{\pi}\frac{d}{dr}\int\limits_0^r \frac{\Gamma[\chi(t)]\,dt}{\sqrt{r^2-t^2}} \tag{4.9}$$

Substituting here the solution of the integral equation (3.2), we obtain another form of the solution of the problem under consideration. If we make use of the solution of the indicated equation in the form of the series (3.4), then it is necessary to compute the finite Hilbert transform of the Jacobi polynomial with the corresponding weight. Such a computation has been performed for the first time in [11]. The corresponding result (for $a = 1$) has the form

$$\Gamma\left[\frac{(1+x)^{\beta}}{(1-x)^{-\alpha}} P_m^{\alpha,\beta}(x)\right] = \pi \operatorname{ctg} \pi\alpha (1-x)^{\alpha} (1+x)^{\beta} P_m^{\alpha,\beta}(x) - \frac{\Gamma(\alpha)\Gamma(\beta+m+1) F(m+1, -\alpha-\beta-m; 1-\alpha; {}^1/_2 - {}^1/_2 x)}{\pi 2^{-\alpha-\beta}\Gamma(\alpha+\beta+m+1)} \qquad (4.10)$$

It is contained also in [6]. In order to obtain the formulas for the contact stresses for a punch with a plane base given in [8], it is necessary to substitute (4.4) into (4.9), take into account (4.10) and perform the limiting process $\nu \to 0$.

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